Worcester County Mathematics League

Varsity Meet 4 – March 5, 2025

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 4 - March 5, 2025
Answer Key

Round 1 - Elementary Number Theory

1. 360
2. 211 or $\{211\}$
3. 18

Round 2 - Algebra I

1. 2
2. $\frac{5}{9}, \frac{14}{9}$ or $\left\{\frac{5}{9}, \frac{14}{9}\right\}$
3. $(-6, -1), (-18, 11)$ need both ordered pairs, either pair first

Round 3 - Geometry

1. 51
2. 9
3. $\sqrt{370}$

Round 4 - Logs, Exponents, and Radicals

1. 4
2. 3
3. $-188, 196$ need both, either order

Round 5 - Trigonometry

1. $150^\circ, 210^\circ$ need both, either order with or without $^\circ$ symbol
2. $\sqrt{13}$
3. $\frac{25}{324}$

Team Round

1. 10201
2. 36
3. 15
4. $2, -1$ need both, either order
5. 10π
6. 1936
7. $\frac{3\sqrt{3}}{2}$
8. 7
9. $6\sqrt{21}$

Worcester County Mathematics League
Varsity Meet 4 - March 5, 2025
Round 1 - Elementary Number Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the least common denominator of the fractions $\dfrac{1}{60}$ and $\dfrac{5}{72}$.

2. Find the set of all prime numbers between 200 and 220.

3. Let S be the set of positive integers n for which the number $\frac{1}{n}$ terminates when written both in base 30 and in base 42. Find the 10^{th} smallest n in S.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. $\{$____________________$\}$

(3 pts) 3. $n =$ ____________________

Algonquin, Worc. Acad., Tahanto/QSC

Worcester County Mathematics League
Varsity Meet 4 - March 5, 2025
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Given whole numbers m and n such that $49 < n < 101$ and $19 < m < 51$, what is the largest possible value of $\dfrac{n+m}{n}$?

2. Solve for x:

$$\left(3x+\frac{1}{3}\right)^2 - 7\left(3x+\frac{1}{3}\right) + 10 = 0$$

3. Find all ordered pairs of real numbers (x, y) that satisfy:

$$|x+y+7| + \left|\frac{1}{2}x + |2-y|\right| = 0$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. $x =$ ____________________

(3 pts) 3. $(x, y) \in \{$ ____________________ $\}$

Burncoat, Bancroft, Westborough/QSC

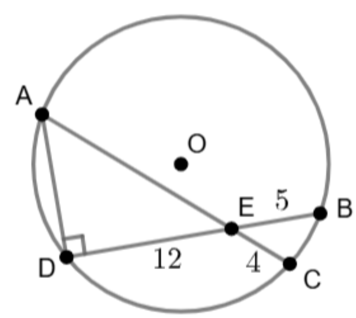
Worcester County Mathematics League
Varsity Meet 4 - March 5, 2025
Round 3 - Geometry

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Let θ be the measure of $\angle A$ in degrees. The measure of the supplement of $\angle A$ is $12°$ greater than three times the measure of the complement of $\angle A$. Find θ.

2. A circular hole 6 inches in diameter is cut in the top of an empty box. A sphere, 10 inches in diameter, is placed so that it rests in the hole without reaching the bottom of the box. What is the farthest distance, measured in inches, from a point on the sphere to the plane of the box top?

3. Given $\odot O$ shown at right, where chords $\overline{AC}$ and $\overline{BD}$ intersect at point E, $\overline{AD} \perp \overline{BD}$, $DE = 12$, $BE = 5$, and $CE = 4$; the circumference of $\odot O$ equals πx. Find x.



ANSWERS

(1 pt) 1. ____________________ °

(2 pts) 2. ____________________ inches

(3 pts) 3. $x =$ ____________________

Shepherd Hill, St. John's, Holliston

Worcester County Mathematics League

Varsity Meet 4 - March 5, 2025

Round 4 - Logs, Exponents, and Radicals

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Solve for x:

$$8^{x+1} = 32^{x-1}$$

2. Find a given:

$$\log_a x = 2\log_x a$$
$$\log_a 3x = 2\log_x a + 1$$

3. Solve for y:

$$\sqrt[3]{9\left(y-4\right)^4} = 2304$$

ANSWERS

(1 pt) 1. $x =$ ____________________

(2 pts) 2. $a =$ ____________________

(3 pts) 3. $y =$ ____________________

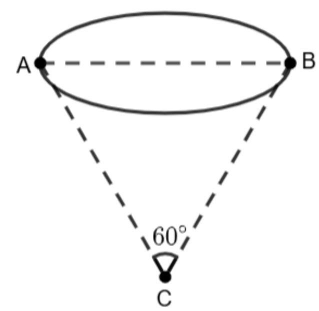
Clinton, Auburn, Shrewsbury

Worcester County Mathematics League
Varsity Meet 4 - March 5, 2025
Round 5 - Trigonometry

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find all values of θ such that $0° \leq \theta < 360°$ and $\cos\theta = -\frac{\sqrt{3}}{2}$.

2. A surveyor needs to measure the width of a lake shown in the figure at right, with AB the width of the lake. Standing at C she measures 4 km to A, 3 km to B and $m\angle ACB = 60°$. Find AB in km.



3. Given that $\sin x + \cos x = \frac{2}{3}$ find:

$$\sin^4 x + \cos^4 x - \left(\sin^6 x + \cos^6 x\right).$$

ANSWERS

(1 pt) 1. $\theta \in \left\{ \underline{\hspace{4cm}} \right\}$

(2 pts) 2. ______________________ km

(3 pts) 3. ______________________

Bromfield, St. John's, Bancroft

Worcester County Mathematics League
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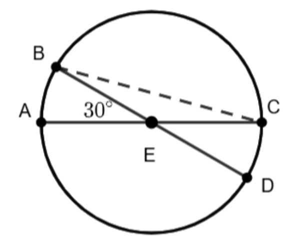
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. A *proper divisor* of positive integer n is any positive integer that divides n (leaving no remainder), other than n itself. Find the greatest *proper divisor* of 1030301.

2. It takes twice as long to adjust each of the three carburators on Monica's classic car as it does to adjust the brakes on any one of the four wheels. If a mechanic spends one hour adjusting all the carburators and brakes, how many minutes, in total, will he spend adjusting all three carburators?

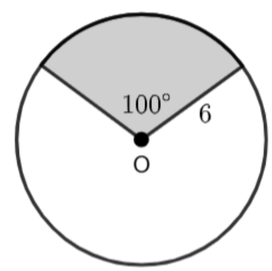
3. $\overline{ED}$ is a radius of the circle shown in the figure at right. The measure of $\angle AEB$ is $30°$. What is the measure of $\angle DBC$, in °?



4. Solve the following equation for x, where x is a Real number:

$$x - x^2 = \log_{\sqrt{2}}\left(\frac{1}{2}\right)$$

5. A circle has radius equal to 6. Find the area of a sector, shaded in the figure at right, with central angle $100°$. Express your answer as a multiple of π.



6. A man was born in the year k^2. He died on his 89th birthday in the year $(k+1)^2$. In what year was he born?

7. Find the exact area of a regular hexagon inscribed in a unit circle. Express your answer in simplest radical form.

8. Solve for x: $$\sqrt{x+9} - \sqrt{x+2} = \sqrt{4x-27}$$

9. The diagonals of a parallelogram have lengths 60 and 48. They intersect at an angle of $60°$. Find the shorter side of the parallelogram.

St. John's,Westborough,Hudson Cath.,Quaboag,Bromfield,South,Notre Dame,St. Peter-Marian,Doherty

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Round 1 - Elementary Number Theory

1. Find the least common denominator of the fractions $\frac{1}{60}$ and $\frac{5}{72}$.

Solution: Recall that the least common denominator (LCD) of a set of fractions is the least common multiple (LCM) of their denominators. In this case the denominators are 60 and 72.

First, find the prime factorizations of the two numbers:

$$60 = 4 \cdot 15 = 2^2 \cdot 3 \cdot 5$$
$$72 = 8 \cdot 9 = 2^3 \cdot 3^2$$

The LCM will contain all of the factors of the two numbers. More particularly, recall that its prime factorization will contain all of the prime factors of the two numbers, with each prime taken to the largest power that appears in the individual prime factorizations.

Compare the exponents of prime factors 2, 3, and 5 in the two prime factorizations: 2^3 is a factor of 72, 3^2 is a factor of 72, and 5 is a factor of 60. Therefore the prime factorization of the LCM/LCD is $2^3 \cdot 3^2 \cdot 5$ and the LCD equals $8 \cdot 9 \cdot 5 = 8 \cdot 5 \cdot 9 = 40 \cdot 9 = \boxed{360}$.

2. Find the set of all prime numbers between 200 and 220.

Solution: Note that there are 19 integers between 200 and 220. Proceed, testing and excluding numbers divisible by the prime numbers $2, 3, 5, 7$, etc. Of those integers, exactly 9 (202, 204, ... 218) are even (divisIble by 2) and therefore not prime.

List the ten remaining odd numbers (201, 203, ... 217, 219) and test for divisibility by primes 3, 5, 7, ... The following numbers are divisible by 3: 201, 207, 213, and 219. Exclude these, leaving 6 numbers. Two more numbers are divisible by 5: 205 and 215. Now $203 = 7 \cdot 29$ and $217 = 7 \cdot 31$ are both divisible by 7, leaving only 209 and 211. Note that $209 = 11 \cdot 19$, which leaves only 211.

Certify that 211 is prime. Test 211 for divisibility by 2, 3, 5, 7, 11, and 13. None of these divides 211 evenly: $211 = 2 \cdot 105 + 1 = 3 \cdot 70 + 1 = 5 \cdot 42 + 1 = 7 \cdot 30 + 1 = 11 \cdot 19 + 2 = 13 \cdot 16 + 3$. Note that $17^2 = 289 > 211$ and if 17 were a factor then there would need be a second factor that is a lower prime. But all the lower primes have been tested, and it is not necessary to test for divisibility by 17 or any other higher prime. The set of prime numbers in the given range is therefore $\boxed{\{211\}}$.

3. Let S be the set of positive integers n for which the number $\frac{1}{n}$ terminates when written both in base 30 and in base 42. Find the 10^{th} smallest n in S.

Solution: First, for $n = 1$, $\frac{1}{n} = \frac{1}{1} = 1$ terminates in any base. Next, consider all $n > 1$ such that $\frac{1}{n}$ terminates in base 10: $n = 2$, also $n = 4, 8, 16$ because $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{8} = 0.125$, $\frac{1}{16} = 0.0625$. Also, $n = 5, 25, 125$ all terminate: $\frac{1}{5} = 0.2$, $\frac{1}{25} = 0.04$, $\frac{1}{125} = 0.008$. In addition, if $n = 2^k \cdot 5^l$ and $k > l$ then $\frac{1}{n} = \frac{1}{2^k \cdot 5^l} = 2^{-l} \cdot 5^{-l} \cdot 2^{l-k} = 10^{-l} \cdot \frac{1}{2^{k-l}}$ will terminate. A similar argument applies if $l \geq k$.

To summarize, $\frac{1}{n}$ will terminate in base 10 for any $n = 2^k \cdot 5^l$ where k and l are Whole numbers, that is, any n whose only prime factors are 2 and/or 5. That is true because $10 = 2 \cdot 5$, that is, 2 and 5 are the only prime factors of 10.

Now, $30 = 2 \cdot 3 \cdot 5$ and $42 = 2 \cdot 3 \cdot 7$. Applying the above logic, $\frac{1}{n}$ will terminate when represented in base 30 if and only if $n = 2^k \cdot 3^l \cdot 5^m$ and it will terminate when represented in base 42 if and only if $n = 2^k \cdot 3^l \cdot 5^q$. It will terminate in both bases if and only if $m = q = 0$, that is, if $n = 2^k \cdot 3^l$.

Choose small values of k and l to generate a ordered series of integers $n = 2^k \cdot 3^l$. For instance:

$$
\begin{aligned}
(k,l) = (0,0) &\implies n = 1 \\
(k,l) = (1,0) &\implies n = 2 \\
(k,l) = (0,1) &\implies n = 3 \\
(k,l) = (2,0) &\implies n = 4 \\
(k,l) = (1,1) &\implies n = 6 \\
(k,l) = (3,0) &\implies n = 8 \\
(k,l) = (0,2) &\implies n = 9 \\
(k,l) = (2,1) &\implies n = 12 \\
(k,l) = (4,0) &\implies n = 16 \\
(k,l) = (1,2) &\implies n = 18 \\
(k,l) = (3,1) &\implies n = 24
\end{aligned}
$$

The tenth n in this ordered sequence is $\boxed{18}$.

Round 2 - Algebra I

1. Given whole numbers m and n such that $49 < n < 101$ and $19 < m < 51$, what is the largest possible value of $\dfrac{n+m}{n}$?

Solution: First, note that
$$\frac{n+m}{n} = \frac{n}{n} + \frac{m}{n} = 1 + \frac{m}{n}.$$
Note that the largest (maximum) value of $1 + \frac{m}{n}$ occurs when $\frac{m}{n}$ has its maximum value. Note also that the maximum value of $\frac{m}{n}$ is reached when m is as large as possible and n is as small as possible.

The largest value of m is 50. The smallest value of n is 50. Then the largest value of $\dfrac{m}{n}$ is
$$\frac{m}{n} = \frac{50}{50} = 1$$
and the largest value of $\frac{n+m}{n} = 1 + \frac{m}{n} = 1 + 1 = \boxed{2}$.

2. Solve for x:
$$\left(3x + \frac{1}{3}\right)^2 - 7\left(3x + \frac{1}{3}\right) + 10 = 0$$

Solution: Let $y = 3x + \frac{1}{3}$, substitute into the given equation, and factor the resulting quadratic:
$$y^2 - 7y + 10 = 0$$
$$(y-2)(y-5) = 0.$$
Then $y = 2$ or $y = 5$. If $y = 3x + \frac{1}{3} = 2$, then $3x = 2 - \frac{1}{3} = \frac{6}{3} - \frac{1}{3} = \frac{5}{3}$ and $x = \frac{1}{3} \cdot \frac{5}{3} = \frac{5}{9}$.

If $y = 3x + \frac{1}{3} = 5$, then $3x = 5 - \frac{1}{3} = \frac{15}{3} - \frac{1}{3} = \frac{14}{3}$ and $x = \frac{1}{3} \cdot \frac{14}{3} = \frac{14}{9}$. Therefore $x = \boxed{\dfrac{5}{9} \text{ or } \dfrac{14}{9}}$.

3. Find all ordered pairs of real numbers (x, y) that satisfy:

$$|x+y+7| + \left|\frac{1}{2}x + |2-y|\right| = 0$$

Solution: First, note that if $|u| + |v| = 0$, then $u = v = 0$. Therefore all solutions satisfy the following system of equations:

$$x + y + 7 = 0$$
$$\frac{1}{2}x + |2-y| = 0$$

Subtract $y+7$ from both sides of the top equation so that $x = -y-7$, and substitute this expression into the bottom equation:

$$\frac{1}{2}(-y-7) + |2-y| = 0$$

Now, $|2-y| = 2-y$ or $|2-y| = y-2$ depending whether $y \geq 2$ or not. Thus, there are two possible solutions for y:

$$\frac{1}{2}(-y-7) + 2 - y = 0 \quad \text{or}$$
$$\frac{1}{2}(-y-7) + y - 2 = 0$$

Multiply the first equation by 2 and solve: $-y-7+4-2y = -3y-3 = 0$, so that $3y = -3$ and $y = -1$.

Similarly, multiply the second equation by 2 and solve: $-y-7+2y-4 = y-11 = 0$ and $y = 11$.

Now find $x = -y-7$ for the two values of y. For $y = -1$, $x = -(-1)-7 = 1-7 = -6$. For $y = 11$, $x = -11-7 = -18$.

Therefore the ordered pairs (x, y) that solve the original equation are $\boxed{(-6, -1) \text{ and } (-18, 11)}$.

Round 3 - Geometry

1. Let θ be the measure of $\angle A$ in degrees. The measure of the supplement of $\angle A$ is 12° greater than three times the measure of the complement of $\angle A$. Find θ.

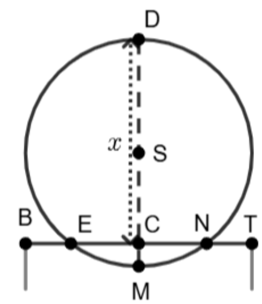
Solution: Recall that if an angle has degree measure θ, then its supplement has measure $180-\theta$ and its complement has measure $90-\theta$. Write the given information in a equation and solve for θ:

$$\begin{aligned} 180-\theta &= 3(90-\theta)+12 \\ &= 270-3\theta+12 \\ 3\theta-\theta &= 270-180+12 = 90+12 \\ 2\theta &= 102 \end{aligned}$$

and $\theta = \frac{102}{2} = \boxed{51}^\circ$.

2. A circular hole 6 inches in diameter is cut in the top of an empty box. A sphere, 10 inches in diameter, is placed so that it rests in the hole without reaching the bottom of the box. What is the farthest distance, measured in inches, from a point on the sphere to the plane of the box top?

Solution: Visualize the top of the box as a (horizontal) plane whose intersection with the sphere is the cutout circle. Next, visualize a vertical plane that is perpendicular to the box top and contains the center of the sphere. It intersects the sphere in a circle with the same radius (10 inches) as the sphere. The figure on the right shows the cross section (of the sphere and box) lying in the vertical plane. $\overline{BT}$ is the intersection of the box top plane and the vertical plane, S is the center of the sphere (and also the center of the cross section circle), and C is the center of the box top circular hole. Draw a line (shown dashed) perpendicular to $\overline{BT}$ through S, intersecting the circle at D and M. Then $\overline{DM}$ is a diameter of $\odot S$ and its length is 10 inches. It intersects $\overline{BT}$ at C. Furthermore, $x = DC$ is the farthest distance from the box top plane to a point on the sphere. Then $CM = DM - DC = 10 - x$.

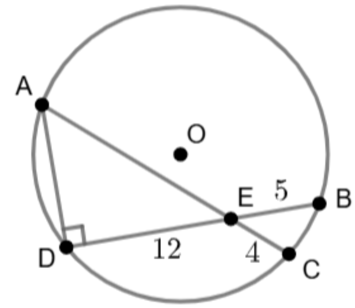


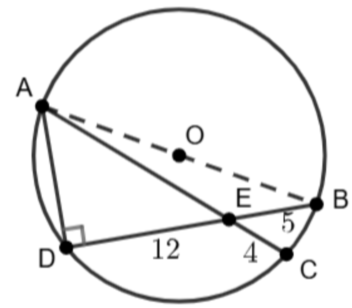
Let E and N be the points of intersection of $\overline{BT}$ and $\odot S$. Then $EC = 3 = NC$ because both $\overline{EC}$ and $\overline{NC}$ are radii of $\odot C$. Recall that when two chords intersect inside the circle, the product of the segment lengths of one chord equals the product of the segment lengths of the other chord. Therefore $EC \cdot NC = CM \cdot CD$. Plug in the specified values, and $3^2 = (10-x)\cdot x$, or $9 = 10x - x^2$. Collect terms on the left hand side and factor the quadratic:

$$\begin{aligned} x^2 - 10x + 9 &= 0 \\ (x-9)(x-1) &= 0 \end{aligned}$$

Then $x = 9$ or $x = 1$. Select the larger distance, and $x = \boxed{9}$ inches.

3. Given $\odot O$ shown at right, where chords $\overline{AC}$ and $\overline{BD}$ intersect at point E, $\overline{AD} \perp \overline{BD}$, $DE = 12$, $BE = 5$, and $CE = 4$; the circumference of $\odot O$ equals πx. Find x.





Solution: First, draw $\overline{AB}$ (shown as the dashed line in the bottom figure at right) and note that inscribed $\angle ADB$ intersects arc AB. Recall that the measure of an inscribed angle is equal to half the measure of its intercepted arc. Since $m\angle ADB = 90°$, the measure of arc AB is twice that, or $180°$. Therefore arc AB is a semicircle, and $\overline{AB}$ is a diameter of $\odot O$. Recall that the circumference of a circle is equal to πd, where d is the diameter of the circle, or AB in this case. It remains to find $AB = x$.

Now $\triangle ADB$ is a right triangle, and the Pythagorean Theorem can be used to find AB if AD and DB are known. $DB = DE + EB = 12 + 5 = 17$. Next, find AE so that the Pythagoreant Theorem can be applied to $\triangle ADE$ to find AD. Recall that if two chords intersect inside a circle, the product of the segment lengths of one chord equals the product of the segment lengths of the other chord. Therefore $EC \cdot AE = EB \cdot DE$, $4 \cdot AE = 5 \cdot 12 = 60$, and $AE = \frac{60}{4} = 15$. Then $AD^2 + DE^2 = AE^2$, or $AD^2 = AE^2 - DE^2 = 15^2 - 12^2 = 225 - 144 = 81 = 9^2$. Therefore $AD = 9$.

Finally, $AB^2 = x^2 = AD^2 + DB^2 = 9^2 + 17^2 = 81 + 289 = 370$, and $x = \boxed{\sqrt{370}}$.

Note that $370 = 10 \cdot 37 = 2 \cdot 5 \cdot 37$, and that 370 has no perfect squares in its factorization. Therefore $\sqrt{370}$ is in simplest radical form.

Round 4 - Logs, Exponents, and Radicals

1. Solve for x:

$$8^{x+1} = 32^{x-1}$$

Solution: First, note that $8 = 2^3$ and $32 = 2^5$. Then change the base of each exponential to 2.

$$\left(2^3\right)^{x+1} = \left(2^5\right)^{x-1}$$
$$2^{3(x+1)} = 2^{5(x-1)}$$

Next, set the two exponents equal and solve for x:

$$3(x+1) = 5(x-1)$$
$$3x + 3 = 5x - 5$$
$$3 + 5 = 5x - 3x = 2x$$

and $2x = 3 + 5 = 8$, so $x = \frac{8}{2} = \boxed{4}$.

2. Find a given:

$$\log_a x = 2\log_x a$$
$$\log_a 3x = 2\log_x a + 1$$

Solution: First, subtract the top equation from the bottom equation and then apply the identity $\log_b x - \log_b y = \log_b \frac{x}{y}$:

$$\log_a 3x - \log_a x = 2\log_x a + 1 - 2\log_x a$$
$$\log_a \frac{3x}{x} = 1$$
$$\log_a 3 = 1$$

Apply the identity $\log_b x = y \implies b^y = x$, and $a^1 = 3$, or $a = \boxed{3}$.

3. Solve for y:

$$\sqrt[3]{9\left(y-4\right)^4} = 2304$$

Solution: First note that 9 is a factor of 2304 because $2304 \div 9 = 256$, so that $2304 = 256 \cdot 9$. Convert this product to a prime factorization: $2304 = 2^8 \cdot 3^2$. Next, replace 2304 by its prime factorization in the original equation, cube both sides, and apply laws of exponents:

$$\begin{aligned}
\sqrt[3]{9\left(y-4\right)^4} &= 2^8 \cdot 3^2 \\
\left(\sqrt[3]{9\left(y-4\right)^4}\right)^3 &= \left(2^8 \cdot 3^2\right)^3 \\
9\left(y-4\right)^4 &= 2^{8\cdot 3} \cdot 3^{2\cdot 3} \\
3^2\left(y-4\right)^4 &= 2^{24} \cdot 3^6
\end{aligned}$$

Next, multiply both sides of the equation by 3^{-2} to isolate $(y-4)^4$ on the left hand side, and express the right hand side as $(\text{something})^4$:

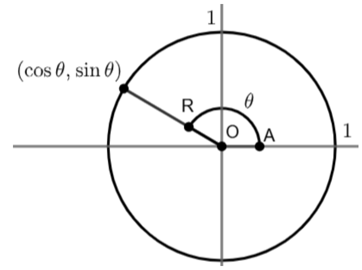
$$\begin{aligned}
3^{-2} \cdot 3^2\left(y-4\right)^4 &= 2^{24} \cdot 3^6 \cdot 3^{-2} \\
3^{-2+2}\left(y-4\right)^4 &= 2^{24} \cdot 3^{6-2} \\
1 \cdot \left(y-4\right)^4 &= 2^{6\cdot 4} \cdot 3^4 \\
\left(y-4\right)^4 &= \left(2^6\right)^4 \cdot 3^4 \\
\left(y-4\right)^4 &= \left(2^6 \cdot 3\right)^4
\end{aligned}$$

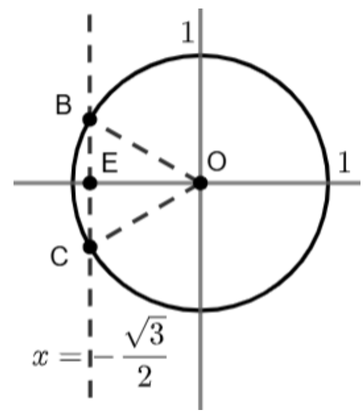
Note that $y-4$ may be positive or negative, so that either $y-4 = 2^6 \cdot 3 = 192$ or $y-4 = -2^6 \cdot 3 = -192$. Add 4 to both sides of both equations, and $y = 192 + 4$ or $y = -192 + 4$. Therefore $y = \boxed{-188 \text{ or } 196}$.

Round 5 - Trigonometry

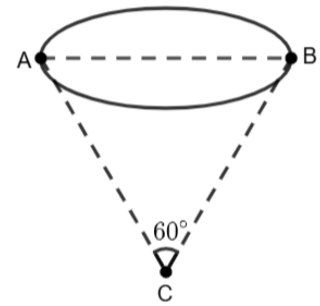
1. Find all values of θ such that $0° \leq \theta < 360°$ and $\cos\theta = -\frac{\sqrt{3}}{2}$.

Solution: Begin by drawing a unit circle, shown at right. Recall that the measure of an arc is equal to the measure of its central angle. Also recall that each point on the unit circle has coordinates $(x, y) = (\cos\theta, \sin\theta)$, where θ is the measure of the arc drawn counterclockwise from the x-axis to (x, y), shown as the arc with central angle $\angle AOR$. Next, note that if $\cos\theta = -\frac{\sqrt{3}}{2}$, then $x = -\frac{\sqrt{3}}{2}$. Recall that if (x, y) is a point on the unit circle, then $x^2 + y^2 = 1$, or $y^2 = 1 - x^2$. Thus $x^2 = \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{(-\sqrt{3})^2}{2^2} = \frac{3}{4}$, and $y^2 = 1 - \frac{3}{4} = \frac{1}{4}$. Thus, $y = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$ and there are two points on the unit circle for which $\cos\theta = -\frac{\sqrt{3}}{2}$: $\left(-\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$, labeled as B and C on the second figure at right.
Draw right triangles $\triangle OBE$ and $\triangle OCE$ and note that each triangle has side lengths of $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and 1 and is therefore a 30°-60°-90° triangle. Thus, $m\angle BOE = m\angle COE = 30°$. The measure of the arc drawn to B is 180° (the measure of a semicircle) minus 30° ($= m\angle BOE$), and the measure of the arc drawn to C is 180° plus 30° ($= m\angle COI$). Thus, the two values of θ are 180 ± 30, or $\boxed{150, 210}°$.





2. A surveyor needs to measure the width of a lake shown in the figure at right, with AB the width of the lake. Standing at C she measures 4 km to A, 3 km to B and $m\angle ACB = 60°$. Find AB in km.



Solution: Recall the Law of Cosines, which states:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where one angle of the trangle has measure θ, the lengths of the sides adjacent to θ are a and b, and the length of the side opposite to θ is c. In this case, $a = 4$, $b = 3$, $c = AB$, and $\theta = 60°$. Therefore:

$$\begin{aligned}(AB)^2 &= 4^2 + 3^2 - 2(4)(3)\cos 60° \\ &= 16 + 9 - 2(12)(\frac{1}{2}) \\ &= 25 - 12 = 13\end{aligned}$$

and $AB = \boxed{\sqrt{13}}$ km.

3. Given that $\sin x + \cos x = \frac{2}{3}$ find:

$$\sin^4 x + \cos^4 x - \left(\sin^6 x + \cos^6 x\right).$$

Solution: Start by squaring the given equation:

$$\begin{aligned}
(\sin x + \cos x)^2 &= \left(\frac{2}{3}\right)^2 \\
\sin^2 x + 2\sin x \cos x + \cos^2 x &= \frac{2^2}{3^2} \\
1 + 2\sin x \cos x &= \frac{4}{9} \\
2\sin x \cos x &= \frac{4}{9} - 1 = -\frac{5}{9} \\
\sin x \cos x &= -\frac{5}{18}
\end{aligned}$$

where the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ was applied to simplify the second equation. Next, square the Pythagorean identity to generate an expression containing $\sin^4 x + \cos^4 x$, applying the binomial identity $(a+b)^2 = a^2 + 2ab + b^2$:

$$\begin{aligned}
1^2 &= \left(\sin^2 x + \cos^2 x\right)^2 \\
1 &= \sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x \\
1 &= \sin^4 x + \cos^4 x + 2\left(\sin x \cos x\right)^2
\end{aligned}$$

so that $\sin^4 x + \cos^4 x = 1 - 2\left(\sin x \cos x\right)^2$.

Next, cube the Pythagorean identity to generate an expression containing $\sin^6 x + \cos^6 x$, applying the binomial identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$:

$$\begin{aligned}
1^3 &= \left(\sin^2 x + \cos^2 x\right)^3 \\
1 &= \sin^6 x + 3\sin^4 x \cos^2 x + 3\sin^2 x \cos^4 x + \cos^6 x \\
1 &= \sin^6 x + \cos^6 x + 3\sin^2 x \cos^2 x \left(\sin^2 x + \cos^2 x\right) \\
1 &= \sin^6 x + \cos^6 x + 3\left(\sin x \cos x\right)^2
\end{aligned}$$

and $\sin^6 x + \cos^6 x = 1 - 3\left(\sin x \cos x\right)^2$. Finish up by substituting for $\sin^4 x + \cos^4 x$ and $\sin^6 x + \cos^6 x$, then substituting $\sin x \cos x = -\frac{5}{18}$:

$$\begin{aligned}
\sin^4 x + \cos^4 x - \left(\sin^6 x + \cos^6 x\right) &= 1 - 2\left(\sin x \cos x\right)^2 - \left(1 - 3\left(\sin x \cos x\right)^2\right) \\
&= 3\left(\sin x \cos x\right)^2 - 2\left(\sin x \cos x\right)^2 = \left(\sin x \cos x\right)^2 \\
&= \left(-\frac{5}{18}\right)^2 = \frac{(-5)^2}{18^2} = \boxed{\frac{25}{324}}
\end{aligned}$$

Team Round

1. A *proper divisor* of positive integer n is any positive integer that divides n (leaving no remainder), other than n itself. Find the greatest *proper divisor* of 1030301.

Solution: First, note that nonzero digits alternate with zero digits in 1030301. Also note the pattern of the nonzero digits: $1, 3, 3, 1$. These are the coefficients of the binomial expansion of a cube. Next, write 1030301 as the sum of powers of 10 and compare this summation with the expansion of $(a+b)^3$:

$$\begin{aligned} 1030301 &= 1000000 + 30000 + 300 + 1 \\ &= 10^6 + 3 \cdot 10^4 + 3 \cdot 10^2 + 1 \\ &= \left(10^2\right)^3 + 3\left(10^2\right)^2 + 3\left(10^2\right) + 1 \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned}$$

Equate the first terms of last two lines and $a^3 = \left(10^2\right)^3 = (100)^3$, so that $a = 100$. Equate the last terms of last two lines and $b^3 = 1$, so that $b = 1$. Check the middle terms: $3a^2b = 3(100)^2 \cdot 1 = 30000$ and $3ab^2 = 3 \cdot 100 \cdot 1^2 = 300$. Therefore $1030301 = (100+1)^3 = (101)^3$.

Note that $2, 3, 5,$ and 7 are not divisors of 101, so 101 is a prime number. Therefore the divisors of 1030301 are $1, 101, 101^2$, and 101^3. The last divisor, $101^3 = 1030301$ is not a proper divisor of 1030301. The greatest proper divisor of 1030301 is therefore $101^2 = \boxed{10201}$.

2. It takes twice as long to adjust each of the three carburators on Monica's classic car as it does to adjust the brakes on any one of the four wheels. If a mechanic spends one hour adjusting all the carburators and brakes, how many minutes, in total, will he spend adjusting all three carburators?

Solution: Let c be the number of minutes that it takes to adjust one carburator and let b be the number of minutes to adjust the brakes on one wheel. Then $c = 2b$. One hour is 60 minutes, so that:

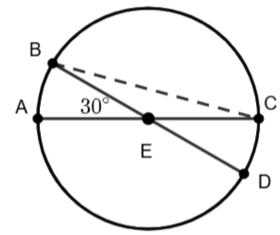
$$3c + 4b = 60$$

Substitute $b = \frac{c}{2}$ into the equation:

$$3c + 4\frac{c}{2} = 3c + 2c = 5c = 60$$

and $c = 60 \div 5 = 12$ minutes. Then $3c = 3 \cdot 12 = \boxed{36}$ minutes.

3. $\overline{ED}$ is a radius of the circle shown in the figure at right. The measure of $\angle AEB$ is 30°. What is the measure of $\angle DBC$?



Solution: Note that $\angle AEB$ and $\angle DEC$ are vertical angles and therefore are congruent. Then m$\angle DBC = 30°$. Recall that the measure of a central angle is equal to the measure of its intercepted arc, so that the measure of arc DC is 30°. Finally, recall that the measure of an inscribed angle is half the measure of its intercepted arc, so that m$\angle DBC = \frac{30}{2} = \boxed{15}°$.

4. Solve the following equation for x, where x is a Real number:

$$x - x^2 = \log_{\sqrt{2}}\left(\frac{1}{2}\right)$$

Solution: First simplify $\log_{\sqrt{2}}\left(\frac{1}{2}\right)$. Recall that $\log_b \frac{x}{y} = \log_b x - \log_b y$ for any base b. Then $\log_{\sqrt{2}}\left(\frac{1}{2}\right) = \log_{\sqrt{2}} 1 - \log_{\sqrt{2}} 2 = -\log_{\sqrt{2}} 2$ because $\log_b 1 = 0$ for any base b. Next, recall the inverse property of logarithms, which states that $\log_b x = y \implies b^y = x$ and let $y = \log_{\sqrt{2}} 2$, so that:

$$\sqrt{2}^y = 2 = \left(\sqrt{2}\right)^2$$

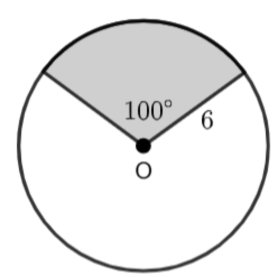
and $y = 2 = \log_{\sqrt{2}} 2$. Then $\log_{\sqrt{2}}\left(\frac{1}{2}\right) = -\log_{\sqrt{2}} 2 = -2$.

Next, replace the right hand side of the original equation with -2, group terms on one side, and factor the quadratic:

$$\begin{aligned} x - x^2 &= -2 \\ 0 &= x^2 - x - 2 \\ 0 &= (x-2)(x+1) \end{aligned}$$

and $x = \boxed{-1 \text{ or } 2}$.

5. A circle has radius equal to 6. Find the area of a sector, shaded in the figure at right, with central angle 100°. Express your answer as a multiple of π.



Solution: Recall that the area of a circle is equal to πr^2, where r is the radius of the circle. Recall also that a sector with central angle measure $\theta°$ has area equal to $\frac{\theta}{360} A_c$, where A_c is the area of the circle. Thus, the area of the sector is $\frac{100}{360}\pi\left(6^2\right) = \frac{10}{36} \cdot 36\pi = \boxed{10\pi}$.

6. A man was born in the year k^2. He died on his 89th birthday in the year $(k+1)^2$. In what year was he born?

Solution: First, find k after noting that the year of his death minus the year of his birth equals 89. Therefore:

$$\begin{aligned}(k+1)^2 - k^2 &= 89\\ k^2 + 2k + 1 - k^2 &= 89\\ 2k + 1 &= 89\end{aligned}$$

and $2k = 89 - 1 = 88$ so that $k = 88 \div 2 = 44$. Now the year of his birth is $44^2 = (40+4)^2 = 40^2 + 2 \cdot 40 \cdot 4 + 4^2 = 1600 + 320 + 16 = \boxed{1936}$.

Note that the man died in the year $1936 + 89 = 2025$: this year!

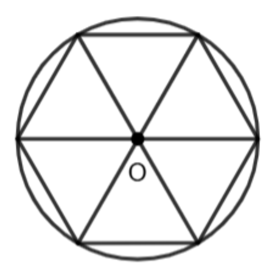
7. Find the exact area of a regular hexagon inscribed in a unit circle. Express your answer in simplest radical form.

Solution: First, draw a regular hexagon inscribed in a circle, shown at right. Note that the center of a regular polygon inscribed in a circle is the center of the circle, and also that the line segments joining the polygon center to the vertices of the polygon are radii of the circle. The six sides of the hexagon are chords of the circle and they intersect six congruent arcs because congruent chords intersect congruent arcs. Each arc is one sixth of the circle and hence measures $360 \div 6 = 60°$, as do each of the six central angles between adjacent radii.

Now consider the six triangles formed by two adjacent radii and a chord. Each triangle is isosceles because the two radii are congruent, and each triangle is equilateral because its vertex angle (between the radii) is $60°$. Thus, the area of the hexagon is equal to six times the area of one of the equilateral triangles. In this case, the triangles have side lengths equal to 1 because the radius of the unit circle is 1.

The area of an equilateral triangle with side length s is equal to $\frac{\sqrt{3}}{4}s^2$, and the area of an equilateral triangle with side length 1 is equal to $\frac{\sqrt{3}}{4}$. The area of the hexagon is equal to 6 times the area of one of the triangles:

$$6 \cdot \frac{\sqrt{3}}{4} = 3 \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{3\sqrt{3}}{2}}.$$



8. Solve for x: $$\sqrt{x+9} - \sqrt{x+2} = \sqrt{4x-27}$$

Solution: Begin by squaring both sides of the equation, then isolate the square root on one side:

$$\left(\sqrt{x+9} - \sqrt{x+2}\right)^2 = \left(\sqrt{4x-27}\right)^2$$
$$\left(\sqrt{x+9}\right)^2 - 2\sqrt{x+9}\sqrt{x+2} + \left(\sqrt{x+2}\right)^2 = 4x - 27$$
$$x + 9 + x + 2 - 2\sqrt{x+9}\sqrt{x+2} = 4x - 27$$
$$-2\sqrt{x+9}\sqrt{x+2} = 4x - 27 - x - 9 - x - 2$$
$$-2\sqrt{(x+9)(x+2)} = 2x - 38$$
$$\sqrt{(x^2+11x+18)} = -\frac{2x-38}{2} = 19 - x$$

Next, square both sides of this equation and solve for x:

$$\left(\sqrt{x^2+11x+18}\right)^2 = (19-x)^2$$
$$x^2 + 11x + 18 = x^2 - 38x + 361$$
$$11x + 38x = 361 - 18$$
$$49x = 343$$

And $x = 343 \div 49 = 7$. The method of squaring both sides of an equation is prone to extraneous solutions. Therefore the final step is to check the solution in the original equation. Substitute $x = 7$ in the left hand side and:

$$\sqrt{x+9} - \sqrt{x+2} = \sqrt{7+9} - \sqrt{7+2}$$
$$= \sqrt{16} - \sqrt{9} = 4 - 3 = 1.$$

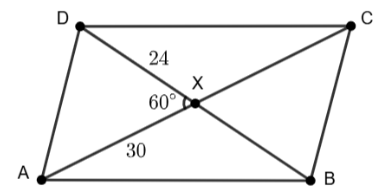
Substitute $x = 7$ in the right hand side and:

$$\sqrt{4 \cdot 7 - 27} = \sqrt{28 - 27} = 1$$

Therefore $x = \boxed{7}$ is the solution to the original equation.

9. The diagonals of a parallelogram have lengths 60 and 48. They intersect at an angle of $60°$. Find the shorter side of the parallelogram.

Solution: First sketch the parallelogram, starting with the two diagonals and minding the fact that the diagonals of a parallelogram bisect each other. Label the vertices of the parallelogram, as shown at right. Note that $\triangle AXD$ has side lengths $DX = 24$ and $AX = 30$ with an included angle of $60°$. Apply the Law of Cosines to this triangle to find AD, the length of the shorter side:



$$\begin{aligned}(AD)^2 &= (AX)^2 + (DX)^2 - 2(AX)(DX)\cos 60° \\ &= 24^2 + 30^2 - 2(24)(30)\frac{1}{2} \\ &= 576 + 900 - 720 = 1476 - 720 = 756\end{aligned}$$

and $AD = \sqrt{756} = \sqrt{9 \cdot 84} = \sqrt{9 \cdot 4 \cdot 21} = \sqrt{36}\sqrt{21} = \boxed{6\sqrt{21}}$.